Construction of curved domain walls

T. Dobrowolski
Institute of Physics AP, Podchorażych 2, 30-084 Cracow, Poland
(Received 3 March 2008; published 28 May 2008)

The evolution of the domain wall is reduced to the dynamics of the membrane. The rigorous construction of curved membranes in the real scalar field models with domain-wall solutions is presented. An example of the curved membrane is discussed in detail. The curved domain-wall solutions are constructed.

DOI: 10.1103/PhysRevE.77.056608 PACS number(s): 05.45.Yv, 11.27.+d, 98.80.Cq

I. INTRODUCTION

Many physical phenomena are described by field-theoretical models that allow the existence of topological solitons. The simplest structures of this type are called kinks (in 1+1 dimensions) or domain walls (in 3+1 dimensions). They appear as solutions of scalar field models with some nonlinear potentials. Domain walls play an important role in the description of many physical systems. In low-energy physics, they have been observed in ferroelectric crystals and in magnetics [1], where they are described in the framework of the Landau-Ginzburg paradigm. They also appear in high-energy physics, i.e., in particle physics, where they describe a quark bag, and in cosmology [2]. The systems with the potentials mentioned above are also generally used in the description of the formation of interfaces and fronts during the phase transition from symmetric to broken symmetry phase in many condensed-matter systems [3].

In the context of relativistic field-theoretical models, there is a long-standing problem concerning the description of the evolution of a single domain wall. The complexity of the problem is a consequence of the spatial extension of the domain wall and the nonlinearity of the underlying field equations. The information on the dynamics of the domain wall comes from numerical and analytical studies as well. The main focus of the analytical studies of the dynamics of a domain wall has been to reduce the field-theoretical description to the simplified effective theory of a classical membrane. One of the most fruitful approaches to the description of the evolution of the domain wall is the improved expansion in width method [4]. This method allows for perturbative description of the evolution of the quite broad class of the domain-wall solutions. In this paper, we concentrate on the exact description of the evolution of the domain-wall solutions, which depend only on the variable perpendicular to the domain-wall surface.

The paper is organized as follows. In Sec. II, we introduce the comoving coordinate system and fix our notation. Section III is devoted to an analysis of the field equations on the domain-wall ansatz. In this section, we reduce the field dynamics to the dynamics of the membrane. We also analyze the form of the integrability conditions of the world hypersurface of the membrane. In Sec. IV, we discuss in detail the particular solution of the membrane equations and show the construction of the curved domain-wall solutions based on the solutions of the membrane equations. The final section contains conclusions.

II. PRELIMINARIES ON THE EMBEDDING OF THE WORLD HYPERSURFACE

We would like to consider the motion of membranes in the models of a real scalar field defined by the Lagrangian

$$L = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - V(\phi), \qquad (1)$$

where we assume that the potential $V(\phi)$ allows the existence of solutions in the form of the domain walls. The corresponding energy-momentum tensor and Euler-Lagrange equation have the form

$$T_{\mu\nu} = \partial_{\mu}\phi\partial_{\nu}\phi - \eta_{\mu\nu}L, \qquad (2)$$

$$\partial_{\mu}\partial^{\mu}\phi + \frac{\delta V}{\delta\phi} = 0. \qquad (3)$$

We assume that the equations of motion have static solutions in the form of domain walls,

$$\phi_W = \phi_W(z), \qquad (4)$$

where the particular z dependence of the solution follows from the reduced equation of motion,

$$-\partial_z^2\phi_W + \left.\frac{\delta V}{\delta\phi}\right|_{\phi=\phi_W} = 0. \qquad (5)$$

The energy density for these solutions is significantly different from zero only in the close vicinity of the X-Y plane. On the other hand, the field ϕ_W disappears, i.e., is equal to zero, exactly on the X-Y plane. In the static case, the Lagrangian (1) can be reduced to the Landau-Ginzburg free energy for ferromagnetic and ferroelectric materials. In these systems, the solution (4) describes the behavior of the order parameter on the board of two domains. In our approach, we identify the surface of zeros of the scalar field with some classical membrane. In this paper, we are interested in the evolution of the above-mentioned membrane, i.e., in the geometry of its world hypersurface Σ. We identify the position of the points of the world hypersurface in the Minkowski space-time by the vector $X^{\mu}(\tau,\sigma^1,\sigma^2)$, where parameters (τ,σ^1,σ^2) are variables on the three-dimensional manifold Σ. We presume that σ^1,σ^2 are spacelike parameters, i.e., they enumerate positions of the points on the membrane surface, and τ is a timelike parameter that describes the evolution of the membrane. For convenience, we introduce more compact notation: $(\sigma^a)=(\sigma^0,\sigma^1,\sigma^2)=(\tau,\sigma^1,\sigma^2)$. Near the world sheet, we introduce the comoving coordinates $(\zeta^\alpha)=(\sigma^a,\xi)$

$$x^\mu = X^\mu(\sigma^a) + \xi n^\mu(\sigma^a), \tag{6}$$

where the vector n^μ is normal to the membrane and ξ is a coordinate in the direction of the vector n^μ. By definition, the spacelike four-vector n^μ is normalized to unity and is orthogonal to the vectors $X^\mu_{,a}$—tangent to the world-hypersurface Σ,

$$n^\mu n_\mu = -1, \quad n_\mu X^\mu_{,a} = 0. \tag{7}$$

The minus sign in the first relation of Eq. (7) is consistent with metric convention $(+,-,-,-)$. The Cartesian coordinates in the laboratory frame are denoted in a standard way, i.e., x^μ. In this paper, we also use the metric induced on the hyper-surface Σ,

$$X^\mu_{,a} X_{\mu,b} \equiv g_{ab}. \tag{8}$$

The embedding of the world hypersurface is described by the extrinsic curvature coefficients K_{ab} and they follow from the Gauss-Weingarten formulas,

$$\partial_a X^\mu_{,b} = X^\mu_{,ab} = \Gamma^c_{ab} X^\mu_{,c} + K_{ab} n^\mu, \tag{9}$$

$$\partial_a n^\mu = n^\mu_{,a} = K^c_a X^\mu_{,c}. \tag{10}$$

In the calculus, we will need the Minkowski space-time metric in the comoving coordinates $(\zeta^\alpha)=(\sigma^a,\xi)$,

$$G_{\alpha\beta} = \frac{\partial x^\mu}{\partial \zeta^\alpha}\frac{\partial x^\nu}{\partial \zeta^\beta}\eta_{\mu\nu}. \tag{11}$$

The explicit form of this metric is the following:

$$G_{\alpha\beta} = \begin{bmatrix} G_{ab} & 0 \\ 0 & -1 \end{bmatrix}, \tag{12}$$

where

$$G_{ab} = S^c_a S_{cb} \tag{13}$$

and

$$S_{ab} \equiv g_{ab} + \xi K_{ab}. \tag{14}$$

The components of the inverse metric in curved coordinates are given by the following formulas:

$$G^{\alpha\beta} = \begin{bmatrix} G^{ab} & 0 \\ 0 & -1 \end{bmatrix}, \tag{15}$$

where

$$G^{ab} = (S^{-1})^a_c (S^{-1})^{bc}, \tag{16}$$

and $(S^{-1})^{ab}$ is inverse to S_{ab}.

III. REDUCTION OF THE FIELD EQUATION TO MEMBRANE CONDITIONS

In the notation introduced in the preceding section, the field Eqs. (3) of the considered models in the comoving coordinates read

$$\frac{1}{\sqrt{-G}}\partial_a(\sqrt{-G}G^{ab}\partial_b\phi) - \frac{1}{\sqrt{-G}}\partial_\xi(\sqrt{-G}\partial_\xi\phi) + \frac{\delta V}{\delta\phi} = 0, \tag{17}$$

where

$$\sqrt{-G} = \sqrt{g}\mathcal{G} \tag{18}$$

and

$$\mathcal{G} = 1 + \xi K^a_a + \frac{1}{2}\xi^2(K^a_a K^b_b - K^a_b K^b_a) + \frac{1}{6}\xi^3(K^a_a K^b_b K^c_c - 3K^a_a K^b_c K^c_b + 2K^a_b K^b_c K^c_a). \tag{19}$$

The explicit form of the metric determinant reduces the field equations in the curved coordinates to the form

$$G^{ab}\partial_a\partial_b\phi + \tfrac{1}{\sqrt{g}}\partial_a(\sqrt{g})G^{ab}\partial_b\phi + \tfrac{1}{\mathcal{G}}\partial_a(\mathcal{G})G^{ab}\partial_b\phi + \partial_a G^{ab}\partial_b\phi - \tfrac{1}{\mathcal{G}}\partial_\xi(\mathcal{G})\partial_\xi\phi - \partial^2_\xi\phi + \tfrac{\delta V}{\delta\phi} = 0. \tag{20}$$

In this paper, we look for possible deformations of the domain-wall solutions, and therefore we assume the analytical form of the functions $\phi=\phi_W(\xi)$ identical with the solutions of Eq. (5), i.e.,

$$-\partial^2_\xi\phi_W + \left.\frac{\delta V}{\delta\phi}\right|_{\phi=\phi_W} = 0. \tag{21}$$

The natural consequence of the adopted form of the field ϕ is a lack of dependence on the variables σ^a, i.e., $\partial_a\phi_W = 0$. The equation of motion (20) for function ϕ_W reduces substantially,

$$-\frac{1}{\mathcal{G}}\partial_\xi(\mathcal{G})\partial_\xi\phi_W = 0. \tag{22}$$

In the preceding equation, the function ϕ_W has fixed shape and the only unknowns are coefficients of the external curvature K_{ab}. In the next step, we expand Eq. (22) with respect to the normal coordinate ξ,

$$(-K^a_a + \xi K^a_b K^b_a - \xi^2 K^a_b K^b_c K^c_a + \cdots)\partial_\xi\phi_W(\xi) = 0. \tag{23}$$

We also assume that the solution of Eq. (5) is regular on the X-Y plane or has poles of finite order. In the first three orders of expansion, we obtain the following constrains:

$$K^a_a = 0, \tag{24}$$

$$K^b_a K^a_b = 0, \tag{25}$$

and

$$K^a_b K^b_c K^c_a = 0. \tag{26}$$

The second order of the expansion was obtained under the assumption that the first order is satisfied, and the third order was obtained under the assumption that the first and the second orders are fulfilled. We do not check the higher orders of expansion because if we use conditions (24)–(26), then we find that $\partial_\xi\mathcal{G}=0$. This result means that Eq. (22) is fulfilled independently of the analytical form of the function ϕ_W.

In addition, we notice that from Eqs. (18) and (19), it follows that conditions (24)–(26) guarantee the global existence of the curvilinear coordinates used by us,

$$\sqrt{-G} = \sqrt{g} \neq 0.$$

Summing up, we have found that the shape of the domain wall and its evolution is exactly and globally described by the following geometrical equations:

$$K^a_a = 0, \quad K^b_a K^a_b = 0, \quad K^a_b K^b_c K^c_a = 0. \tag{27}$$

The world hypersurface has to obey additional conditions that guarantee its integrability. These conditions follow from the requirement of commutativity of the derivatives ($\partial_a\partial_b n^\mu = \partial_b\partial_a n^\mu$, $\partial_a\partial_b X^\mu_{,c} = \partial_b\partial_a X^\mu_{,c}$) and they have the form

$$R_{abcd} = K_{ad}K_{bc} - K_{ac}K_{bd}, \tag{28}$$

$$\nabla_a K_{bc} = \nabla_b K_{ac}. \tag{29}$$

The first condition, called the Gauss condition (28), establishes an explicit connection between the external and internal geometry of the manifold Σ described by the components of the Riemann curvature tensor,

$$R_{abcd} = \partial_c\Gamma_{bda} - \partial_d\Gamma_{bca} + \Gamma^e_{bc}\Gamma_{ade} - \Gamma^e_{ac}\Gamma_{bde}, \tag{30}$$

where

$$\Gamma_{abc} = g_{ce}\Gamma^e_{ab} \tag{31}$$

and Γ^e_{ab} are Christoffel symbols. One can easily check that the manifold defined by Eqs. (27) corresponds to the world hypersurfaces, with the curvature scalar equal to zero,

$$R = K^b_a K^a_b - K^a_a K^b_b = 0. \tag{32}$$

The Peterson-Codazzi condition (29) contains the covariant derivatives with respect to the internal metric on the manifold Σ, and therefore they also connect its external and internal geometry.

IV. EXAMPLE OF THE CURVED MEMBRANE

Let us consider the hypersurface defined by the vector

$$X^\mu = [\tau, \sigma^1, \sigma^2, \psi(\tau - v^1\sigma^1 - v^2\sigma^2)], \tag{33}$$

where $c=1$ and ψ is three times the integrable function of its arguments. We also assume that constants v^1 and v^2 are not independent and they satisfy the relation $(v^1)^2+(v^2)^2=1$. The proposed hypersurface describes a deformed membrane with a deformation of arbitrary shape ψ propagating along the membrane with the speed of light. Our purpose is to check whether the proposed hypersurface (33) satisfies membrane Eqs. (27) and integrability conditions (28) and (29). First we introduce the variable $u=\tau-v^1\sigma^1-v^2\sigma^2$. One can easily notice that $\partial_\tau u=1$ and $\partial_{\sigma^i} u=-v^i$, where $i=1,2$. In order to calculate the metric induced on the manifold Σ, we need the tangent vectors to the world hypersurface,

$$X^\mu_{,0} = [1,0,0,\psi'], \quad X^\mu_{,1} = [0,1,0,-v^1\psi'],$$

$$X^\mu_{,2} = [0,0,1,-v^2\psi'], \tag{34}$$

where a prime denotes the derivative with respect to the variable u. The metric coefficients have the form

$$g_{ab} = \eta_{ab} - u_a u_b f, \quad g^{ab} = \eta^{ab} + v^a v^b f, \tag{35}$$

where we introduced the following notation: $v^a=(1,v^i)$, $u_a=(1,-v^i)$, $f=\psi'^2$, and η_{ab} is the Minkowski 2+1 metric, i.e., $\eta_{00}=1$, $\eta_{ij}=-\delta_{ij}$, and $\eta_{0i}=0$ $(i,j=1,2)$. The orthogonality and normalization conditions (7) uniquely define the normal vector to the hypersurface Σ,

$$n^\mu = [\psi', v^1\psi', v^2\psi', 1]. \tag{36}$$

The external curvatures follow from Gauss formula (9) projected onto the normal direction,

$$K_{ab} = -n_\mu X^\mu_{,ab}. \tag{37}$$

For hypersurface (33) considered by us, the external curvature coefficients read

$$K_{ab} = u_a u_b \psi'', \tag{38}$$

where we noticed that for arbitrary function χ of the variable u, we have $\partial_a\chi=\chi'\partial_a u=\chi' u_a$. Now we are prepared to check whether Eqs. (27) are fulfilled by the hypersurface (33). First, we notice that the quantities u_a and v^a introduced by us satisfy the following relations: $u_a v^a = 1-(v^1)^2-(v^2)^2=0$, $\eta^{ab}u_b=v^a$, and $\eta^{ab}u_a u_b = v^b u_b=0$. Then we become convinced that the first relation is fulfilled independently of the form of the function ψ,

$$K^a_a = g^{ab}K_{ab} = [\eta^{ab}u_a u_b + (u_a v^a)(u_b v^b)f]\psi'' = 0. \tag{39}$$

In order to check the second relation, we calculate the mixed components of the extrinsic curvature tensor $K^a_b = g^{ac}K_{cb} = v^a u_b \psi''$. Now the second relation,

$$K^a_b K^b_a = v^a u_b v^b u_a (\psi'')^2 = 0, \tag{40}$$

and the third relation,

$$K^a_b K^b_c K^c_a = v^a u_b v^b u_c v^c u_a (\psi'')^3 = 0, \tag{41}$$

are fulfilled trivially, i.e., without any additional conditions on function ψ.

The hypersurface (33) must also satisfy the integrability conditions. We start from Peterson-Codazzi condition (29),

$$\partial_a K_{bc} - \Gamma^e_{ac}K_{be} = \partial_b K_{ac} - \Gamma^e_{bc}K_{ae}. \tag{42}$$

Explicit calculation of the partial derivative of the external curvatures $\partial_c K_{ab} = u_a u_b u_c \psi^{1/2}$ shows that the derivatives in the above formula cancel each other, $\partial_a K_{bc} - \partial_b K_{ac}=0$, and Eq. (42) reduces to the form

$$\Gamma^e_{ac}K_{be} = \Gamma^e_{bc}K_{ae}. \tag{43}$$

Next we calculate the Christoffel symbols for the metric (35) induced on the world hypersurface $\Gamma^c_{ab} = \frac{1}{2}v^c u_a u_b f'$. Finally, we substitute the calculated Γ^c_{ab} and the external curvatures (38) in the last equation, and we can conclude that the Peterson-Codazzi condition is fulfilled,

$$\Gamma^e_{ac}K^i_{be} - \Gamma^e_{bc}K^i_{ae} = \frac{1}{2}v^e u_a u_c f' u_b u_e \psi'' - \frac{1}{2}v^e u_b u_c f' u_a u_e \psi'' = 0. \tag{44}$$

In order to check whether the surface (33) satisfies the Gauss condition, we calculate the following combination of the Christoffel symbols:

$$\Gamma^e_{ad}\Gamma^d_{bc} = \frac{1}{2}v^e u_a u_d f' \frac{1}{2}v^d u_b u_c f' = 0.$$

We also calculate the derivatives of Γ^c_{ab},

$$\partial_d \Gamma^c_{ab} = \frac{1}{2}v^c u_a u_b u_d f''. \quad (45)$$

In the next step, we find out that the Riemann curvatures of the considered surface are zero,

$$R^e_{cab} = \partial_a \Gamma^e_{bc} - \partial_b \Gamma^e_{cb} + \Gamma^e_{ad}\Gamma^d_{bc} - \Gamma^e_{bd}\Gamma^d_{ac} = 0. \quad (46)$$

In the Gauss condition, the curvatures for the hypersurface considered by us appear in the form $R_{abcd}=0$. We verify that the value of the combination,

$$K_{ad}K_{bc} - K_{ac}K_{bd} = u_a u_d \psi'' u_b u_c \psi'' - u_a u_b \psi'' u_b u_d \psi'' = 0, \quad (47)$$

of the external curvatures is zero and therefore the Gauss condition is also fulfilled. At first glance, it seems surprising that all of the coefficients of the Riemann tensor in the world hypersurface (33) are equal to zero. This result means that the domain-wall hypersurface Σ is internally flat like a sheet of paper. On the other hand, the nonzero external curvature coefficients (38) mean that this hypersurface is nontrivially embedded in the Minkowski space-time, which can be compared to a deformed (bent) sheet of paper with deformation traveling at the speed of light.

Finally, we use the surface being considered in order to find the explicit form of the domain-wall solutions that correspond to it. At the beginning, we assumed the existence of the domain-wall solution,

$$\phi_W(z) \rightarrow \phi_W(\xi), \quad (48)$$

with well-defined analytical form in the comoving coordinates. The connection between the comoving and Cartesian laboratory coordinates is given by Eq. (6),

$$z = x^3 = X^3(\sigma^a) + \xi n^3(\sigma^a), \quad (49)$$

where the vector X^μ is given in formula (33) and the normal vector n^μ is written in Eq. (36). The connection between the normal comoving coordinate ξ and the z coordinate has the explicit form

$$z = \psi(\tau - v^1\sigma^1 - v^2\sigma^2) + \xi. \quad (50)$$

Finally, we can write the vortex solutions that correspond to the considered example of the word hypersurface,

$$\phi_W(\xi) = \phi_W[z - \psi(\tau - v^1\sigma^1 - v^2\sigma^2)]. \quad (51)$$

We can also consider particular models that are defined by the special forms of the potentials.

Example 1. The well-known example of the model that possesses domain-wall solutions is the ϕ^4 model defined by the potential $V(\phi)=\frac{1}{4}\lambda(\phi^2-a)^2$. This model is often used in the description of condensed-matter systems and also in particle physics. The analytical form of the domain-wall solution is given by the hyperbolic tangent,

$$\phi_W(z) = \sqrt{a}\tanh\left(\sqrt{\frac{\lambda a}{2}}z\right),$$

and is the special case (for $m=1$) of the more general solution given by the Jacobi elliptic function,

$$\phi_{\rm sol} = \sqrt{\frac{2ma}{m+1}}\,{\rm sn}\left(\sqrt{\frac{\lambda a}{m+1}}z, m\right).$$

The solution defined in the considered model by the hypersurface (33) has the following form [5]:

$$\phi_W = \sqrt{a}\tanh\left(\sqrt{\frac{\lambda a}{2}}[z - \psi(\tau - v^1\sigma^1 - v^2\sigma^2)]\right).$$

Example 2. The other model is used in some contexts of particle physics and is defined by the potential $V(\phi)=\frac{1}{2}-|\phi|+\frac{1}{2}\phi^2$ [6]. The domain-wall solution in this model has the form [7]

$$\phi_W(z) = \pm 2\frac{\tanh\left(\frac{1}{2}z\right)}{1+\tanh\left(\frac{1}{2}|z|\right)}.$$

The solution defined by the hypersurface (33) is the following:

$$\phi_W(z) = \pm 2\frac{\tanh\left\{\frac{1}{2}[z - \psi(\tau - v^1\sigma^1 - v^2\sigma^2)]\right\}}{1+\tanh\left[\frac{1}{2}|z - \psi(\tau - v^1\sigma^1 - v^2\sigma^2)|\right]}.$$

V. CONCLUSIONS

In the current paper, we have proven an exact correspondence between the curved domain-wall solutions and the evolution of the domain-wall world hypersurface Σ, defined as a trace in the Minkowski space-time, of the surface of zeros of the scalar field ϕ. The conditions (27) allow the existence of the deformed domain-wall solutions that have an analytical form that is identical to the known straight domain walls. The membrane equations obtained in Sec. III describe the Nambu-Goto membrane with some additional constrains. The coordinates used by us are well defined globally and therefore the conditions (27) describe global properties of the domain-wall evolution.

In order to show the existence of nontrivial solutions of the conditions (27), we analyzed a particular example of the world hypersurface X^μ that describes the arbitrary shape deformation traveling along the domain wall at the speed of light. We also showed that the considered hypersurface Σ satisfies all integrability conditions. We proved that the internal geometry of this particular hypersurface Σ is trivial, i.e., it has geometry similar to that of the sheet of paper nontrivially embedded in Euclidean space.

Finally, we demonstrated how the hypersurface X^μ can be used to construct the new, deformed domain-wall solutions. We would like to stress that the conclusions of this paper concern domain-wall solutions that depend only on the normal variable to the domain-wall surface.

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